Listing all the minimal separators of a 3-connected planar graph

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Abstract

We present an efficient algorithm that lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Key words: minimal separator; planar graphs; enumeration

1 Introduction

In the last ten years, minimal separators have been increasingly studied in graph theory leading to many algorithmic applications [5,9,10,12].

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [5], Bodlaender *et al.* conjecture that for a class of graphs with a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the concept of potential maximal clique [2] and showed that, if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can be solved in polynomial time. They later showed [3] that if a graph has a polynomial number of minimal separators, it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Extensive research has been performed to compute the set of the minimal separators of a graph [1,6,7,11]. Berry *et al.* [1] proposed an algorithm of running time $O(nm)$ per separator [1] that uses the concept of generating new minimal

[1] The authors only proved a running time of $O(n^3)$ but the actual bound is $O(nm)$ [8].

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separators from a previous minimal separator S by finding the minimal separators contained in $S \cup N(x)$ for $x \in S$. This simple process can generate all the minimal separators of a graph. However, by using this algorithm a minimal separator can be generated many times.

The aim of this article is to address the problem of finding the minimal separators of a 3-connected planar graph G. In order to avoid the problem of recalculation, we define the set $\mathcal{S}_a(S, O)$ of the a, b-minimal separators S' for some b such that the connected component of a in $G \backslash S'$ contains the connected component of a in $G \backslash S$ but avoids the set O. Therefore, it is possible to ensure that a given minimal separator is never computed more than five times.

2 Definitions

Throughout this paper, $G = (V, E)$ is a 3-connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x, y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C,\ (x, y) \in E\}$. When the sets A and B are disjoint, their union is denoted by $A \sqcup B$.

A set $S \subseteq V$ is a *separator* if $G \backslash S$ has at least two connected components, an *a, b-separator* if a and b are in different connected components of $G \backslash S$, an *a, b-minimal separator* if no proper subset of S is an a, b-separator. The connected component of a in $G \backslash S$ is $C_a(S)$. The component $C_a(S)$ *is a full connected component* if $N\Big(C_a(S)\Big) = S$. For an a, b-minimal separator S, both $C_a(S)$ and $C_b(S)$ are full. A set S is a *minimal separator* if there exist a and b such that S is an a, b-minimal separator or, which is equivalent, if it has at least two full connected components. An *$a, *$-minimal separator* of a graph $G = (V, E)$ is an a, b-minimal separator of G for some $b \in V$. The set of the $a, *$-minimal separators is denoted by $\mathcal{S}_a$ and the set of the minimal separators of G is denoted by $\mathcal{S}(G)$.

It is possible to order the $a, *$-minimal separators in the following way:

$$S_1 \preccurlyeq_a S_2 \text{ if } C_a(S_1) \subseteq C_a(S_2).$$

The minimal separator S_1 is *closer* to a than S_2. The set of a, b-minimal separators is a lattice for the relation $\preccurlyeq_a$[4] but we only need the following weaker lemma:

Lemma 1 *Let C be a set of vertices of a graph G inducing a connected subgraph of G, a be a vertex of C and b be a vertex of $G \backslash \Big(C \cup N(C)\Big)$.*

The neighbour S of $C_b\Big(C \cup N(C)\Big)$ is an a,b-minimal separator such that C is a subset of $C_a(S)$ that is closer to a than any a,b-minimal separator S' such that C is a subset of $C_a(S')$.

PROOF. By construction, C is a subset of $C_a(S)$. By definition, the component $C_b(S)$ is full and since S is a subset of $N(C)$, the component $C_a(S)$ is also a full component which implies that S is an a,b-minimal separator.

Let S' be an a,b-minimal separator such that C is a subset of $C_a(S')$. Let p be a path in $C_b(S')$ with b as one of its ends. The vertices of S' are at least at distance 1 of C so the vertices of p are at least at distance 2 of C. Since S is a subset of $N(C)$, $p \cap S = \emptyset$. In other words p is a subset of $C_b(S)$ and $C_b(S') \subseteq C_b(S)$. This last inclusion implies that $C_a(S) \subseteq C_a(S')$ *i.e.* S is closer to a than S'. $\square$

For S an $a,*$-minimal separator and $O \subseteq V$, the set $\mathcal{S}_a(S,O)$ is *the set of the $a,*$-minimal separators S' further from a than S and such that $O \cap C_a(S') = \emptyset$.* If $x \in V$, the set $\mathcal{S}_a^x(S,O)$ is *the set of $S' \in \mathcal{S}_a(S,O)$ such that $x \in C_a(S')$.*

Remark 2 *If $x \in S$, then $\mathcal{S}_a(S,O)$ is the disjoint union*

$$\mathcal{S}_a\Big(S, O \cup \{x\}\Big) \bigsqcup \mathcal{S}_a^x(S,O).$$

More precisely, if $(S_i)_{i \in I}$ are the elements of $\mathcal{S}_a^x(S,O)$ closest to a, then

$$\mathcal{S}_a(S,O) = \mathcal{S}_a\Big(S, O \cup \{x\}\Big) \bigsqcup \Big(\bigcup_{i \in I} \mathcal{S}_a(S_i,O)\Big).$$

This gives the skeleton of an algorithm to compute the set $\mathcal{S}_a(S,O)$.

Remark 3 *If S belongs to $\mathcal{S}_a^x(S,O)$, then $\mathcal{S}_a^x(S,O) = \mathcal{S}_a(S,O)$.*

The algorithm is based on remarks 2 and 3. To list $\mathcal{S}_a$, the algorithm computes the sets $\mathcal{S}_a(S,\emptyset)$ for every S closest to a in $\mathcal{S}_a$. During this calculation, it computes $\mathcal{S}_a(S,O)$ with $O \subseteq S$. To do so, it chooses x in $S \backslash O$ and calculates $\mathcal{S}_a^x(S,O)$ and $\mathcal{S}_a(S, O \cup \{x\})$. The set $\mathcal{S}_a^x(S,O)$ is itself a union of $\mathcal{S}_a(S_i,O)$. But to obtain such a decomposition, one needs to find the elements of $\mathcal{S}_a^x(S,O)$ closest to a, which the following proposition does.

Proposition 4 *Let $G = (V,E)$ be a graph, S an $a,*$-minimal separator, $O \subset S$, $x \in S \backslash O$ and $C = C_a(S) \cup \{x\}$*

The elements of $\mathcal{S}_a^x(S,O)$ closest to a are exactly the neighbourhoods of the connected components of $G \backslash \Big\{N(C) \cup C\Big\}$ that contain O and that are maximal for inclusion.

PROOF. Let S_1 be an a,b-minimal separator of $\mathcal{S}_a^x(S,O)$ closest to a. Let S' be the neighbourhood of $C_b\Big(N(C)\cup C\Big)$. By lemma 1, S' is an a,b-minimal separator such that C is a subset of $C_a(S')$ and S' is closer to a that S_1. Moreover, since $C_a(S_1)\cap O=\emptyset$ and S' is closer to a than S_1, $C_a(S')\cap O\subseteq C_a(S_1)\cap O=\emptyset$. Thus S' belongs to $\mathcal{S}_a^x(S,O)$ and is closer to a than S_1. This proves that $S_1=S'$. Since S_1 cannot be a subset of another element of $\mathcal{S}_a^x(S,O)$, S_1 is the neighbourhood of a connected component of $G\backslash\Big\{N(C)\cup C\Big\}$ which is maximal for inclusion.

Conversely, let S_1 be a neighbourhood of a connected component D of $G\backslash\Big\{N(C)\cup C\Big\}$ that contains O and that is maximal for inclusion. By lemma 1, S_1 is an element of $\mathcal{S}_a^x(S,O)$ that is closer to a than any a,b-minimal separator of $\mathcal{S}_a^x(S,O)$ with b in D. So if S_2 is an a,b-minimal separator of $\mathcal{S}_a^x(S,O)$ strictly closer to a than S_1, S_1 is not an a,b-minimal separator. Suppose for a contradiction that such an a,b-minimal separator exists. It follows from the first part of the proof that such an a,b-minimal separator S_2 closest to a is the neighbourhood of $C_b\Big(N(C)\cup C\Big)$. The set S_2 is an element of $\mathcal{S}_a^x(S,O)$ that is closer to a than S_1 and S_1 is a subset of S_2 (because $S_1\setminus S_2\subseteq C_a(S_2)\setminus C_a(S_1)$ and S_2 is closer to a then S_1) and therefore S_1 is a strict subset of S_2 contradicting the fact that S_1 is maximal for inclusion. $\square$

Proposition 4 gives us a way to find the minimal elements of $\mathcal{S}_a^x(S,O)$, for example by using a graph search to compute the neighbourhoods of the connected components of $G\backslash\{N(C)\cup C\}$ and then choosing among the minimal separators found the ones that contain O and that are maximal by inclusion. Using the skeleton of remark 2, we can construct an algorithm to compute the set $\mathcal{S}_a(S,O)$ that may look like:

Algorithm 1 `_calc3_`

begin
  if $S\backslash O=\emptyset$ **then**
    return$(\{S\})$
  else
    let $x\in S\backslash O$
    $\mathcal{S}\leftarrow$`_calc3_`$(G,a,S,O\cup\{x\})$

    for each S_i *in* `find_closest_elements`(G,a,x,S,O)
      $\mathcal{S}\leftarrow\mathcal{S}\cup$`_calc3_`$(G,a,S_i,O)$
  return$(\mathcal{S})$
end

However several problems need to be solved.

i. We do not know whether the sets $\mathcal{S}_a(S_i, O)$ are disjoint or not. If not, a minimal separator could be computed many times, which would lead to a bad complexity.
ii. To implement the function `find_closest_elements`, proposition 4 states that we can start with a graph search of G.

  But if $\mathcal{S}_a(S, O) = \{S\}$, the recursive calls to the algorithm will try to find an element of $\mathcal{S}_a^x(S, O)$ closest to a for every $x \in S\backslash O$. Each call to `find_min_elements` costs at least $O(m)$ and finally, we would have spent at least $O(nm)$ to realise that $\mathcal{S}_a(S, O) = \{S\}$.

Proposition 6 in section 3.1 ensures that for 3-connected planar graphs, problem (i) is true, *i.e.* if S_1 and S_2 are two minimal elements of $\mathcal{S}_a^x(S, O)$, the sets $\mathcal{S}_a(S_1, O)$ and $\mathcal{S}_a(S_2, O)$ are disjoint. Section 3.3 then shows how to determine whether $\mathcal{S}_a^x(S, O)$ is empty or not in an overall $O(n)$.

3 Planar graphs

In this section, we will consider 3-connected planar graphs without loops.

Let Σ be the plane. A *plane graph* $G_\Sigma = (V_\Sigma, E_\Sigma)$ is a graph drawn on the plane, that is $V_\Sigma \subset \Sigma$ and each $e \in E_\Sigma$ is a simple curve of Σ between two vertices of V_Σ in such a way that the interiors of two distinct edges do not meet. We will denote by $\widetilde{G}_\Sigma$ the drawing of G_Σ. A *planar graph* is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A *face* of G_Σ is a connected component of $\Sigma\backslash\widetilde{G}_\Sigma$.

3.1 Minimal separators of 3-connected planar graphs

Proposition 5 *In a 3-connected planar graph, minimal separators are minimal for inclusion.*

PROOF. Suppose that $S \subset S'$ are two minimal separators of a 3-connected planar graph.

Let a, b, c and d be vertices such that S' is an a,b-minimal separator and S is a c,d-minimal separator. Since S is not an a,b-minimal separator, either $C_c(S')$ or $C_d(S')$ is disjoint with $C_a(S')$ and $C_b(S')$. Suppose that $C_c(S')$ is such a component. In this case, $C_c(S)$ and $N\Big(C_c(S)\Big)$ are respectively equal to $C_c(S')$ and S.

But then G admits $K_{3,3}$ as a minor for if we contract $C_a(S')$, $C_b(S')$ and $C_c(S')$ into the vertices a', b' and c', all these vertices have S in their neighbourhood and since G is 3-connected, $|S| \geq 3$. This contradicts the fact that G is planar. $\square$

Proposition 6 *Let $G = (V, E)$ be a 3-connected planar graph, a a vertex of G, S an $a,*$-minimal separator, O a subset of S and x a vertex of $S \backslash O$.*

If S_1 and S_2 are two distinct elements of $\mathcal{S}_a^x(S, O)$ that are closest to a, then

$$\mathcal{S}_a(S_1, O) \cap \mathcal{S}_a(S_2, O) = \emptyset.$$

PROOF. Let C be $C_a(S) \cup \{x\}$ and suppose for a contradiction that S_3 is a minimal separator of $\mathcal{S}_a(S_1, O) \cap \mathcal{S}_a(S_2, O)$ with S_1 and S_2 two distinct elements of $\mathcal{S}_a^x(S, O)$ closest to a. Let b be a vertex such that S_3 is an a,b-minimal separator.

Since S_3 is further from a than S_1 and S_2, both S_1 and S_2 are a,b-separators. There exists an a,b-minimal separator S' included in S_1. By proposition 5, a minimal separator of G is minimal for inclusion which proves that $S_1 = S'$ and S_1 is an a,b-minimal separator. By lemma 1, the neighbourhood S_4 of $C_b\Big(N(C) \cup C\Big)$ is an a,b-minimal separator such that C is a subset of $C_a(S_4)$ that is closer to a than S_1. So $C_a(S_4) \cap O \subseteq C_a(S_1) \cap O = \emptyset$, and S_4 is an element of $\mathcal{S}_a^x(S, O)$ that is closer to a than S_1. Similarly, S_2 is an a,b-minimal separator and S_4 is closer to a than S_2 which contradicts the fact that S_1 and S_2 are two distinct elements of $\mathcal{S}_a^x(S, O)$ closest to a. $\square$

3.2 The intermediate graph

Definition 7 *Let $G_\Sigma = (V_\Sigma, E_\Sigma)$ be a 3-connected plane graph. Let F be the set of its faces. In each face $f \in F$ pick up one point v_f. Let R_F be the set $\{v_f \mid f \in F\}$. The intermediate graph $G_I = (V_I, E_I)$ is a plane graph whose vertex set is $V_I = V_\Sigma \cup R_F$. We place an edge between a vertex $v \in V$ and $v_f \in R_F$ if and only if the vertex v is incident to the face f.*

For G' a subgraph of G_I, the set $\widetilde{G}' \cap V_\Sigma$ will be denoted by $V(G')$.

Proposition 8 *Let μ be a cycle of G_I such that the curve $\widetilde{\mu}$ separates at least two vertices a and b of V_Σ.*

The set $V(\mu)$ is an a,b-separator of G_Σ.

PROOF. Let p be a path in G_Σ from a to b. Since a and b are not in the same connected component of $\Sigma\backslash\widetilde{\mu}$, $\widetilde{p}$ intersect $\widetilde{\mu}$. By construction, $p \cap \mu \subseteq V_\Sigma$. This implies that every path from a to b meets $V(\mu)$ and so $V(\mu)$ is an a, b-separator. $\square$

Proposition 9 *Let S be an a, b-minimal separator of G. There exists a simple cycle μ of G_I such that the Jordan curve defined by μ separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.*

PROOF. Let C be the connected component of a in $G\backslash S$. Let us contract C into a supervertex v_C to build the graph $G_{/C}$. There is a cycle $\mu_{/C}$ in $(G_{/C})_I$ such that $V(\mu_{/C})$ is the neighbourhood of v_C in $G_{/C}$. Therefore, the neighbourhood of C in G_I has the structure of a cycle μ.

Suppose $\widetilde{\mu}$ is not a Jordan curve, the border μ' of the connected component of b in $\Sigma\backslash\widetilde{\mu}$ is a strict sub-lace of $\widetilde{\mu}$ which separates a and b. However, proposition 8 shows that $V(\mu')$ which is a strict subset of S is an a, b-separator. This contradicts the fact that S is a a, b-minimal separator. $\square$

Proposition 9 shows that the minimal separators of a 3-connected planar graph correspond to cycles of the intermediate graph. Thus, when a set corresponds to no cycle of the G_I, it is not a minimal separator. However, this is not a characterisation of the minimal separators of a 3-connected planar graph for some cycles of G_I correspond to no minimal separator of G.

There are several ways to find an exact criterion for minimal separators. The following section presents a criterion that is well suited to our purpose.

3.3 Ordered separators

Definition 10 *An ordered separator of G is a sequence of distinct vertices*

$$(v_0, \dots, v_{p-1}),\ (p > 2)$$

such that

i. there exists a face to which v_i and $v_{i+1\,[p]}$ are incident;
ii. v_i and v_j are incident to a common face only if $i = j+1\,[p]$ or $j = i+1\,[p]$;
iii. if $p = 3$, no face is incident to v_i, $v_{i+1\,[p]}$ and $v_{i+2\,[p]}$.

The notation $i\,[p]$ means i modulo p.

A set $S = \{v_0, \dots, v_{p-1}\}$ is an ordered separator if there exists a permutation σ such that $(v_{\sigma(0)}, \dots, v_{\sigma(p-1)})$ is an ordered separator.

If $S = (v_0, \ldots, v_{p-1})$ is an ordered separator of G, then S is naturally associated to the set $\{v_0, \ldots, v_{p-1}\}$. We will use an ordered separator either as a sequence or as the corresponding set.

Lemma 11 *Every minimal separator S of G is ordered.*

PROOF. Let S be an a, b-minimal separator of G.

Proposition 9 states that there exists a simple cycle of G_I

$$\mu = (v_0, f_0, \ldots, v_{p-1}, f_{p-1})$$

such that $V(\mu) = S$.

Let us prove that $T = (v_0, \ldots, v_{p-1})$ is an ordered separator corresponding to S.

i. The construction of T ensures that v_i and v_{i+1} are incident to a common face (f_i).
ii. Suppose that v_i et v_j are incident to a common face f and that $i + 1 \neq j\,[\,p\,]$ and $j + 1 \neq i\,[\,p\,]$.
   $\mu_1 = (v_i, f_i, v_{i+1}, f_{i+1}, \ldots, v_j, f)$ and $\mu_2 = (v_j, f_j, v_{j+1}, f_{j+1}, \ldots, v_i, f)$ are laces of G_I. Moreover, since either μ_1 or μ_2 separates a and b, there exists an a, b-separator strictly included in S which is absurd.
iii. Suppose that $p = 3$ and that v_0, v_1 et v_2 are all incident to a common face f. If we add a vertex f to G connected to the vertices v_0, v_1 and v_2, the graph remains planar which is absurd because this graph has $K_{3,3}$ as a minor. Indeed, the connected component of a, the connected component of b and the vertex f are all incident to v_0, v_1 and v_2 which builds up a $K_{3,3}$.

The sequence T is an ordered separator corresponding to S. $\square$

Conversely,

Lemma 12 *Every ordered separator of G is a minimal separator of G.*

PROOF. Let $S = (v_0, \ldots, v_{p-1})$ be an ordered separator of G.

First, S is a separator. Otherwise, $G \backslash S$ would be connected or empty. In both cases, all the vertices of S would be incident to a common face.

Let S' be a minimal separator included in S. By lemma 11, S' is ordered and since condition ii forbids an ordered separator to have a strictly included ordered separator, $S' = S$. The ordered separator S is a minimal separator. $\square$

From lemmata 11 and 12, we have the following proposition:

Proposition 13 *A set $S \subseteq V$ is a minimal separator of a 3-connected planar graph $G = (V, E)$ if and only if it corresponds to an ordered separator of G.*

At this point, we have a characterisation of the minimal separators of a 3-connected planar graph. Let us see how it enables us to find out whether $\mathcal{S}_a^x(S, O)$ is empty or not ($O \subseteq S$ and $x \in S \backslash O$).

Proposition 14 *Let $S = (v_0, \ldots, v_{p-1})$ be an ordered $a, *$-separator of a 3-connected planar graph $G = (V, E)$ and $O = (v_0, \ldots, v_i)$, $(i < p - 1)$ be an initial sequence of S.*

If there exists a face that is incident to both $y \in N(v_{i+1}) \backslash C_a(S)$ and v_j with $0 < j < i$, then $\mathcal{S}_a^{v_{i+1}}(S, O)$ is empty.

PROOF. Let b be such that S is an a, b-minimal separator and suppose that $y \in N(v_{i+1})$ and v_j with $0 < j < i$ are both incident to a face f. Since S is an ordered separator, there exists a cycle $(v_0, f_0, \ldots, v_k, f_k)$ of G_I corresponding to a Jordan curve $\widetilde{\mu}$. Let Σ_b be the connected component of $\Sigma \backslash \widetilde{\mu}$ that contains b. Since y and v_j are incident to f, there exists a path (v_{i+1}, g, y, f, v_j) that corresponds to a curve $\widetilde{\nu}$ that cuts Σ_b in two parts Σ_b^1 and Σ_b^2 whose borders are $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ respectively. Since $0 < j < i$, neither $V(\widetilde{\mu}_1)$ nor $V(\widetilde{\mu}_2)$ contains O.

Suppose that S' is an element of $\mathcal{S}_a^{v_{i+1}}(S, O)$ closest to a. Let c be such that S' is an a, c-minimal separator. The vertex c belongs to Σ_b. We may suppose that c belongs to Σ_b^1. By proposition 4, S' is the neighbourhood of $C_c\big(S \cup N(v_{i+1})\big)$ *i.e.* $S' = V(\widetilde{\mu}_1)$, but O is not a subset of S' which is absurd. $\square$

Conversely,

Proposition 15 *Let $S = (v_0, \ldots, v_{p-1})$ be an ordered $a, *$-separator of a 3-connected planar graph $G = (V, E)$ and $O = (v_0, \ldots, v_i)$, $(i < p - 1)$ be an initial sequence of S.*

If there is no face incident to both $y \in N(v_{i+1}) \backslash C_a(S)$ and v_j $(0 < j < i)$, then there is an ordered separator in $S \cup N(v_{i+1}) \backslash C_a(S)$ that contains O.

PROOF. The neighbours $(y_1, \ldots, y_l)$ of v_{i+1} taken in clockwise order are such that y_i and y_{i+1} are incident to a common face. Moreover, since v_{i+1} and v_i are both incident to a face f_1 and since v_{i+1} and v_{i+2} are both incident to a face f_2, there is a sequence $P = (v_i, x_1, \ldots, x_k, v_0)$ such that there exists a face incident to any two consecutive vertices of P and such that P uses only vertices of $N(v_{i+1})\backslash C_a(S)$ and $v_{i+2}, \ldots, v_{p-1}$. One such sequence is

$$(v_i, y_j, y_{j+1}, \ldots, y_k, v_{i+2}, \ldots, v_{p-1}, v_0).$$

Let P be such a sequence between v_i and v_0 of minimal length. Together with $(v_1, \ldots, v_{i-1})$, P forms an ordered separator of G as required. □

4 The algorithm

The properties of the previous section allow us to build up an algorithm to compute the set $\mathcal{S}_a(S, O)$ with $O \subseteq S$.

Algorithm 2 `calc3_aux`

input:

G a 3-connected planar graph
a a vertex of G
$S = (v_0, \ldots, v_{p-1})$ an ordered separator such that $a \notin S$
$O = (v_0, \ldots, v_i)$ with $i \leq p-1$ a subset of S
The vertices that have an incident face in common with v_l $(l \neq 0)$ are tagged l unless they can be tagged j $(1 \leq j \leq l-1)$.
These vertices are the forbidden vertices.
The vertices of $C_a(S)$ are also tagged "$C_a(S)$".

output:

$\mathcal{S}_a(S, O)$ the set of a, b-minimal separators S' further from a than S such that $C_a(S') \cap O = \emptyset$.

begin
    if $i = p-1$ **then**
        return($\{S\}$)
    else
        $x \leftarrow v_{i+1}$
        $\mathcal{S} \leftarrow$`calc3_aux`$(G, a, S, (v_0, \ldots, v_i, x))$

        for each $y \in N(x)$ *not tagged "$C_a(S)$"*
            if y *is tagged* $j < i$ **then**
                return($\mathcal{S}$)
        for each S' *in* `find_closest_elements`(G, a, x, S, O)
            tag the vertices according to S'
            $\mathcal{S} \leftarrow \mathcal{S} \cup$ `calc3_aux`$(G, a, S', (v_0, \ldots, v_i))$
end

Proposition 16 *The algorithm* `calc3_aux` *is correct. It computes the set*

$\mathcal{S}_a(S,O)$ *of a 3-connected planar graph.*

PROOF. The algorithm is an application of remark 2 and proposition 13, 14 and 15. $\square$

Proposition 17 *The algorithm can be implemented to compute the set* $\mathcal{S}_a(S,O)$ *in time* $O(n|\mathcal{S}_a(S,O)|)$.

PROOF. The algorithm `_calc_3_aux` is a recursive version of the **for** loop below:

> **for** l **from** $i+1$ **to** $p-1$
>   `empty`$\leftarrow$`FALSE`
>   **for each** $y \in N(v_l)$ not tagged "$C_a(S)$"
>   **if** y is tagged $j < l-1$ **then**
>   `empty`$\leftarrow$`TRUE`
>   **if** not `empty` **then**
>   **for each** S' in `find_closest_elements`$(G, a, v_l, S, (v_0, \ldots, l-1))$
>   tag the vertices according to S'
> $\mathcal{S} \leftarrow \mathcal{S} \cup$ `calc3_aux`$(G, a, S', (v_0, \ldots, l))$
> **return**$(\mathcal{S})$

For each minimal separator S, the algorithm performs the following operations:

i. the function `find_closest_elements` produces S;
ii. the vertices of G are tagged;
iii. the **for** loop is executed in the recursive call to `calc3_aux`
iv. S is returned.

The function `find_closest_elements` can be implemented in linear time. Computing the neighbourhoods of the connected component of $G \backslash \{N(C) \cup C\}$ that contain O can clearly be done in linear time with a graph search, but not computing those that are maximal for inclusion. However, since the graph is 3-connected planar, anyone of these neighbourhoods is necessarily maximal for inclusion, because if some neighbourhood S was a strict subset of some other neighbourhood S' then S' would be a minimal separator that is not minimal for inclusion, which would contradict proposition 5. Another graph search can be used to tag all the vertices. This costs $O(n+m)$.

The **for** loop tests the neighbours of v_l to check if they are forbidden. Since the vertex v_l is always different, this costs at most $O(m)$.

In a planar graph, the number m of edges satisfies $0 \leq m \leq 3n-6$, so the time spent on each minimal separator is $O(n)$, which gives an overall time complexity of $O(n|\mathcal{S}_a(S,O)|)$. $\square$

The following algorithm uses the function `calc3_aux` to compute the set of all minimal separators of a planar graph G.

Algorithm 3 `all_min_sep3`

input:
    G a 3-connected planar graph
output:
    the set of $a,*$-minimal separators of G

begin
    $\mathcal{S} \leftarrow \emptyset$
    find $a \in V$ `with` $d(a) < 6$
    for each *minimal separator* $S \subseteq N(a)$
        $\mathcal{S} \leftarrow \mathcal{S} \cup$ `calc3_aux`$(G, a, S, \emptyset)$
    for each $y \in N(a)$
        for each $a,*$-*minimal separator* $S \subseteq N(y)$
            $\mathcal{S} \leftarrow \mathcal{S} \cup$ `calc3_aux`$(G, y, S, \emptyset)$
    return$(\mathcal{S})$
end

Theorem 18 *Algorithm* `all_min_sep3` *computes the set of the minimal separators of a 3-connected planar graph in time $O(n|\mathcal{S}(G)|)$*

PROOF. Since in a 3-connected planar graph minimal separators are minimal for inclusion, given a vertex a, $S \in \mathcal{S}(G)$ either belongs to $\mathcal{S}_a$ or runs through a. In the second case, it is a b,$*$-minimal separator for a neighbour b of a.

Moreover, there exists a vertex a of degree at most five in a planar graph. Let $b_1, \dots, b_p$ be its neighbours.

By computing $\mathcal{S}_a \bigcup \left(\bigcup_{i\in[1..p]} \mathcal{S}_{b_i}\right)$, a minimal separator can be calculated no more than five times, which gives the claimed complexity. $\square$

5 Conclusion

This article confirms the feeling of Berry *et al.* [1]. In their conclusion, they note that their algorithm may compute a minimal separator up to n times and that this could be improved. This is exactly what we have gained for 3-connected planar graphs. Our algorithm can be modified to list the minimal separators of an arbitrary planar graph. We also feel that there could be a better general algorithm to compute the minimal separators of a graph.

This article gives another proof that planar graphs and their minimal separators in particular are peculiar. We feel that topological properties such as proposition 9 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

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